Single-Crossing, Strategic Voting and the Median Choice Rule

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Abstract

This paper studies the strategic foundations of the Representative Voter Theorem (Rothstein, 1991), also called the "second version" of the Median Voter Theorem. As a by-product, it also considers the existence of non-trivial strategy-proof social choice functions over the domain of single-crossing preference profiles. The main result presented here is that single-crossing preferences constitute a domain restriction on the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof and group-strategic-proof not only over the full set of alternatives, but also over every possible policy *agenda*. The paper also shows the close relation between single-crossing and order-restriction. And it uses this relation together with the strategy-proofness of the median choice rule to prove that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

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1 Introduction

In the last twenty five years, *single-crossing* has become a "popular" feature of preferences within the field of Political Economy.[1] From the seminal works of Roberts (1977) and Grandmont (1978) and, more recently, due to the theoretical developments of Rothstein (1990, 1991), Gans and Smart (1996) and Austen-Smith and Banks (1999), it is now well-known that this domain restriction is sufficient to guarantee the existence of equilibria in one-dimensional models of majority voting, especially in situations where single-peakedness may not hold.

Moreover, this restriction is not only technically convenient, but it also makes sense in many political settings. In few words, the single-crossing property used in the context of voting, which is similar to that used in principal-agent literature and monotone comparative statics, says that, given any two policies, one of them more to the right than the other, the more rightist is an individual (with respect to another individual) the more he will prefer the right-wing policy over the left-wing one.

Thus, unlike single-peakedness, single-crossing is a restriction that imposes limitations *across* individual preferences, on the character of voters' heterogeneity, rather than on the shape of individual preferences. The main idea behind it is that, in many circumstances, *ordering* people according to a single parameter (like income, productivity, intertemporal preferences, ideological position, etc.) may be more natural than ordering alternatives. Hence, under this condition, the conflict of interests among individuals is assumed to be projected into a one-dimensional parameter space, and then the *types* of the agents are assigned a position over this left-right scale with the requirement that, for any pair of alternatives, the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

It turns out that this condition not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In fact, the core is simply the set of ideal points of the median type *agent* in the ordering of individuals with respect to which the preference profile is single-crossing.[2] This result is sometimes referred to in the literature as the *Representative Voter Theorem* (Rothstein, 1991) (henceforth RVT) or, alternatively, as "the second version" of the Median Voter Theorem (Myerson, 1996 and Gans and Smart, 1996).

[1]See, for example, the different applications found in Persson and Tabellini (2000).

[2]In contrast, under single-peaked preferences, the core of the majority rule consists of the median *ideal points* in the ordering of alternatives with respect to which the profile is single-peaked.

The main problem with this result is that, unlike the *original* Median Voter Theorem over single-peaked preferences, whose non-cooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere voting*. Clearly, this assumption is difficult to maintain in applications that focus on policy choices made in strategic frameworks. Hence, a natural question arises respect to its applicability in those models.

This paper studies the strategic foundation of the Representative Voter Theorem. As a by-product, it also considers the existence of non-trivial strategy-proof social choice functions on the domain of single-crossing preference profiles and over the non-negative real line. There are several reasons that justify to carry out this analysis. But the first and more important one is that, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. In particular, people uses the "single-crossing version" of the Median Voter Theorem without caring much about its strategic foundation. So, one of the main purposes here is to fill out this gap.

In addition, the study is also motivated by a more technical fact, though not less important. The analysis of strategic voting in the context of single-crossing preferences leads to consider strategy-proofness over a preference domain where there exists a linear ordering of the types of the agents and, therefore, a specific kind of *correlation* among individual preferences. This contrasts with much of the work developed in the field, which focuses on social choice rules defined over Cartesian preference domains.[3] Moreover, this feature looks interesting for studying manipulation in multi-dimensional choice spaces and over constrained sets of alternatives, a problem that is extremely important in Political Economy (since voters usually have to choose from sets with only a few policies, rather than from the full set of alternatives).

The main result of the paper shows that single-crossing preferences constitute a domain restriction in the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof and group-strategic-proof not only over the full set of alternatives, but also over every possible policy *agenda*. This paper also shows the close relation between single-crossing and order-restriction. And it uses this relation together with the strategy-proofness of the median choice rule to prove that the collective outcome predicted by the Representative Voter Theorem can be implemented

[3]We thank an anonymous referee for pointing out this issue.

in dominant strategies through a simple mechanism in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

The paper is organized as follows. Section 2 presents the model, the notation and the definitions. Section 3 exhibits the equivalence between single-crossing and *order-restriction* for preferences indexed by the types of the agents. Section 4 presents the non-strategic version of the Representative Voter Theorem (the "order-restricted version" of the Median Voter Theorem). The results related to strategy-proofness and the *indirect* implementation of the median choice rule over single-crossing preferences are presented in section 5, which also uses these and the results of section 3 to derive, as a by-product, the game-theoretic counterpart of the Representative Voter Theorem. The consequences of these results and further lines of research that stem from them are discussed in section 6.

2 The model, notation and definitions

The basic model of single-crossing preferences assumes that the set of agents I is finite and its cardinality $|I| = n > 2$ is odd. Individuals in I must choose a policy (for example, the level of a public good) from a feasible set of alternatives. They do this by voting.

The set of all possible collective outcomes $X = \{x_1, \dots, x_l\}$, $|X| > 2$, is assumed to be a finite subset of the non-negative real line $\Re_+$. The set X is such that $x_j \leq x_k$ for $j \leq k$, where the linear order $\leq$ is the usual order on $\Re_+$. For a vector $x = (x_1, \dots, x_n) \in \Re^n_+$, we let $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $(\hat{x}_i, x_{-i}) = (x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n)$, where $\hat{x}_i \in \Re_+$. In addition, for any group of agents $D \subseteq I$, we denote $(x_D, x_{D^c}) = ((x_i)_{i \in D}, (x_j)_{j \in D^c})$, where $D^c = I \backslash D$.

The set of all feasible alternatives may be either the entire X or just one of its non-empty subsets. The set $\tilde{X}$ represents a generic subset - with the induced order - of X. We use $A(X)$ to represent the set of all non-empty subsets of X, $A(X) = \{\tilde{X} : \tilde{X} \in 2^X \backslash \emptyset\}$. In words, X is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a $\tilde{X} \in A(X)$.

Let $P(X)$ be the set of all complete, transitive and antisymmetric binary orderings of X. We say $P(X)$ is the *universal domain* of individual preferences.[4] Agent i's preferences over the alternatives in X are assumed to be completely characterized by a single parameter $\theta_i \in \Theta = \{\theta^1, \dots, \theta^m\}$, where $\Theta \subset \Re$ is a finite and *ordered* subset of the real line, such that

[4]Indifference between alternatives is not allowed. This is a natural assumption when the set of alternatives is finite.

$\theta^1 < \theta^2 < \ldots < \theta^m$ and $m \leq |P(X)|$. As usual, we interpret θ_i as being agent i's *type*.

That is, we assume there exists a function $\Phi : \Theta \to P(X)$ that assigns a unique element $\succ_{\theta_i} \in P(X)$ to each $\theta_i \in \Theta$. We say that $\succ_i$ represents the preferences of an agent i of type θ_i if,

$$\forall x,\, y \in X,\; x \succ_i y \;\Leftrightarrow\; x\, \Phi(\theta_i)\, y.$$

The following example illustrates how these preferences can arise naturally in many political-economic settings:

Example 1 *(Persson and Tabellini, 2000) Consider the following simplified version of the redistributive distortionary taxation model of Roberts (1977). Suppose individual $i \in I$ has preferences $w(c_i, l_i) = c_i + v(l_i)$, $v'(\cdot) > 0$, $v''(\cdot) \leq 0$, where c_i denotes individual consumption and l_i leisure. The individual's budget constraint is $c_i \leq (1-t)h_i + f$, where $0 < t < 1$ is an income tax rate, f represents a lump-sum transfer and h_i is the individual labor supply. Individuals are heterogenous in a productivity parameter $\theta_i \in \Theta \subset \Re$, which is distributed in the population with mean $\bar{\theta}$. Given these different productivities, each individual i faces an "effective" time constraint $1 - \theta_i \geq l_i + h_i$. Finally, it is assumed that the government runs a balanced budget; i.e., $f \leq t\left(\sum_i h_i/n\right)$. Solving the model, we have that the induced policy preferences of agent i over alternative tax rates are*

$$u_i(t) = u(t\,; \theta_i) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta}),$$

where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply. $\square$

The maximal set associated with the pair $\langle X, \succ_i \rangle$ is $M(X, \succ_i) = \{x \in X : \forall y \in X \backslash \{x\},\, x \succ_i y\}$. That is, $M(X, \succ_i)$ yields the alternative that is top-ranked in X for i with respect to her preferences $\succ_i$. Notice that since preferences are strict, maximal sets are indeed singletons.

A preference profile associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$ is an n-tuple $(\succ_1, \ldots, \succ_n) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ in $P(X)^n$. This means the profile of individual preferences depends on the *state* $\theta \in \Theta^n$: in the state θ, agent i has preferences $\Phi(\theta_i)$ over the set X. This formulation allows for any degree of correlation across the agents' preferences. We assume each agent observes θ, so there is complete information among the agents about their preferences over X. Extending our earlier conventions to preference profiles, we have $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. Similarly, the profile obtained by changing agent i's preferences for $\hat{\succ}_i$ is $(\hat{\succ}_i, \succ_{-i}) = (\succ_1, \ldots, \succ_{i-1}, \hat{\succ}_i, \succ_{i+1}, \ldots, \succ_n)$. Finally, for any group of agents $D \subseteq I$, $(\succ_D, \succ_{D^c}) = ((\succ_i)_{i \in D}, (\succ_j)_{j \in D^c})$.

Now, we restrict the set of admissible preference profiles by imposing a condition on preferences that involves the entire profile:

Definition 1 *A preference profile $(\succ_1, \ldots, \succ_n)$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on X if, for all x, $y \in X$ and all i, $j \in I$ such that $y > x$ and $\theta_j > \theta_i$,*

$$y \, \Phi(\theta_i) \, x \quad \Rightarrow \quad y \, \Phi(\theta_j) \, x. \qquad SC$$

We denote $SC(X)$ the set of all single-crossing preference profiles on X.[5] The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness,[6] single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both domain conditions are independent, in the sense that neither property is logically implied by the other. In Example 1, for instance, it is easy to see that the profile of induced policy preferences $(u_1, \ldots, u_n)$ satisfies single-crossing. However, for $h(t)$ sufficiently convex, it violates single-peakedness. (See also Examples 2 and 3 below.)

Furthermore, from the perspective of the analysis of strategy-proofness, there is a huge difference among these two preference domains. While single-peaked profiles of individual preferences define a subset of $P(X)^n$ that constitutes a Cartesian product, single-crossing profiles do not. That is, $SC(X)$ cannot be written as a Cartesian-product preference domain. The reason is that individual preference orderings (or types) in $(\succ_1, \ldots, \succ_n) \in SC(X)$ are correlated, in the sense specified in Definition 1, instead of being completely independent of each other.

As we will see, this implies that, even if a social choice function (yet to be defined) is strategy-proof on $SC(X)$, a mechanism implementing it has to be more complex than a straightforward one. We will return to this point in the last section of the paper. For the moment, let us illustrate how these preferences look like through the following two examples:

Example 2 *Suppose there are three types (each of them possibly associated to a group of individuals), indexed $\theta_1 < \theta_2 < \theta_3$, who must choose an alternative from the finite subset $\{x, y, z\} \subset \Re_+$, $x < y < z$. Assume that the types have the preferences depicted in Table 1 below. It is easy to see that*

[5] Other expressions used in the literature to denominate this preference restriction are *hierarchical adherence*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Rothstein (1990, 1991), Gans and Smart (1996), Austen-Smith and Banks (1999) and List (2001), and the references quoted there.

[6] Formally, a preference profile $(\succ_1, \ldots, \succ_n)$ is single-peaked on X with respect to the linear order $\leq$ if for all $i \in I$, there exists $\tau_i \in X$, called the *peak* of i associated to the preference relation $\succ_i$, such that (1) $\tau_i \succ_i x$, for all $x \in X \backslash \{\tau_i\}$; (2) $y < x \leq \tau_i$ implies $x \succ_i y$, and (3) $\tau_i \leq x < y$ implies $x \succ_i y$.

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
y	z	y
z	y	x

Table 1: Example 2

this profile is single-crossing on $\{x, y, z\}$. *However, for any ordering of the alternatives, the profile violates single-peakedness.* $\square$

Example 3 *Suppose three individuals,* 1, 2 *and* 3, *that have to choose an alternative from the subset* $\{a, b, c, d\} \subset \Re_+$. *Assume their preferences* $\succ = (\succ_1, \succ_2, \succ_3)$ *are as in Table 2. Then, the profile* $\succ$ *is single-peaked with respect to the ordering of the alternatives* $c < a < b < d$. *However, if each individual* i *is associated to a type* θ_i, *it violates single-crossing.* $\square$

$\succ_1$	$\succ_2$	$\succ_3$
a	d	b
b	b	a
d	a	c
c	c	d

Table 2: Example 3

In the political arena, single-crossing makes sense if, for example, individual types are interpreted as being different ideological characters, arranged in the left-right scale, and the alternatives as public policies to be chosen by the society. Put in this way, it says that, given any two policies, one of them more to the right than the other, the more rightist a type the more will he prefer the right-wing policy over the left-wing one.[7]

[7]Notice the difference with single-peakedness: "Intuitively, a single-peaked profile is one in which the set of alternatives can be ordered along a left-right scale in such a way that each individual has a unique most-preferred alternative (or *ideal point*) and the individual's ranking of other alternatives falls as one moves away from her ideal point. Such profiles capture the common intuition that, for example, an individual has a most preferred ideological position on some liberal-conservative spectrum and the more distant is a candidate's ideological position from this most-preferred point the more the individual dislikes the candidate." (Austen-Smith and Banks (1999), pp. 93.)

Given a preference $\succ_i$ in the profile $\succ \in SC(X)$, we define agent i's *induced* preferences over the agenda $\tilde{X} \in A(X)$, $\tilde{\succ}_i$, as follows:

$$\forall x,\, y \in \tilde{X},\; x \,\tilde{\succ}_i\, y \;\Leftrightarrow\; x \succ_i y.$$

Notice that the property of being single-crossing is preserved in the induced preferences. That is, if $\succ \in SC(X)$ then $\tilde{\succ} \in SC(\tilde{X})$, for all $\tilde{X} \in A(X)$.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity cannot be ensured.

The aggregation process is represented by a social choice function. For any $\tilde{X} \in A(X)$, a *social choice function* f on $SC(\tilde{X})$ is a single-value mapping $f : SC(\tilde{X}) \rightarrow \tilde{X}$ that associates to each preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ a unique outcome $f(\tilde{\succ}) \in \tilde{X}$.

We are primarily interested in aggregation procedures conducted by pairwise majority voting. This rule leads in the domain of single-crossing preferences and under the assumption of *sincere* voting to a collective outcome that coincides with the median type agent's most-preferred alternative (see Theorem 1 below). We will examine in the next sections if agents, endowed with this kind of preferences, have incentives to misrepresent their types in the aggregation process. But first, we need to define some additional concepts.

For any odd positive integer k, let $m^k : \Re^k_+ \rightarrow \Re_+$ be the *k-median function*, defined in the following way: for all $x \in \Re^k_+$, $m^k(x)$ is the *k-median* of $x = (x_1, \ldots, x_k)$ if and only if $|\{x_i \in \Re_+ : x_i \leq m^k(x)\}| \geq \frac{(k+1)}{2}$ and $|\{x_j \in \Re_+ : m^k(x) \leq x_j\}| \geq \frac{(k+1)}{2}$. Because k is odd, this function is always well-defined.

Now, we define the *median choice rule* in the following way. For any individual ordering $\tilde{\succ}_i$ in $\tilde{\succ} \in SC(\tilde{X})$, let $\tau(\tilde{\succ}_i) = M(\tilde{X}, \tilde{\succ}_i)$:

Definition 2 *A social choice function f^m on $SC(\tilde{X})$ is called the median choice rule if for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f^m(\tilde{\succ}) = m^n(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)).$$

A crucial property we seek in a social choice function is *strategy-proofness*, and the related concept of *group-strategy-proofness*. That is, we want to consider voting rules where agents, acting individually or in groups, never have the incentives to misrepresent their preferences. To capture this idea, we define the following two concepts:

Definition 3 *A social choice function f on $SC(\tilde{X})$ is strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for any agent $i \in I$, with type θ_i, any misrepresentation $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $\hat{\theta}_i \neq \theta_i$, is such that either $f(\tilde{\succ}) \,\tilde{\succ}_i\, f(\hat{\succ}_i, \tilde{\succ}_{-i})$ or $f(\tilde{\succ}) = f(\hat{\succ}_i, \tilde{\succ}_{-i})$, where $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$.*[8]

If a social choice function f is not strategy-proof, then there exist $i \in I$ and $\hat{\succ}_i$ such that for some $\tilde{\succ}_{-i}$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and i's true preferences, $\tilde{\succ}_i$, $f(\hat{\succ}_i, \tilde{\succ}_{-i}) \,\tilde{\succ}_i\, f(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then, we say f is *manipulable* at $(\tilde{\succ}_i, \tilde{\succ}_{-i})$, by i, via $\hat{\succ}_i$. In the same way:

Definition 4 *A social choice function f on $SC(\tilde{X})$ is group-strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for every coalition $D \subseteq I$, with types $\theta_D = (\theta_i)_{i \in D}$, there does not exist a joint misrepresentation $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, $\hat{\theta}_D \neq \theta_D$, such that, for all $i \in D$, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \,\tilde{\succ}_i\, f(\tilde{\succ})$, where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$.*

In the following sections, we will study how well the median choice rule performs, according to these manipulation criteria, on the domain of single-crossing preference profiles. But, since the main motivation to do this is to study the strategic foundation of the Representative Voter Theorem (the "single-crossing version" of the Median Voter Theorem), let us discuss first the connection between single-crossing and order-restriction, which is the original domain where this Theorem was formulated.

3 Single-crossing and order-restriction

Order-restriction, introduced formally for the first time by Rothstein (1990, 1991), is a preference restriction that has been shown to be closely related to single-crossing (Gans and Smart, 1996). Next we provide its definition and an *equivalence* theorem (up to renaming of types) that parallels that result, but that is more consistent with Rothstein's original characterization.[9]

For any two sets of integers A and B, let $A >_S B$, read "A is higher than B", if for every $a \in A$ and $b \in B$, $a > b$.

Definition 5 *A preference profile $(\Phi(\theta_1), \ldots, \Phi(\theta_n)) \in P(X)^n$ is order-restricted on X if and only if there exists a permutation $\gamma : \Theta \to \Theta$ such that for all distinct pair of alternatives $x, y \in X$, either*

$$\{\gamma(\theta) \in \Theta : x\, \Phi(\gamma(\theta))\, y\} >_S \{\gamma(\theta) \in \Theta : y\, \Phi(\gamma(\theta))\, x\} \qquad OR-1$$

[8]With $\tilde{\Phi}(\cdot)$ we represent the restriction of $\Phi(\cdot)$ over $\tilde{X}$.

[9]In this section, we will make definitions and proofs over X, but everything is equally valid for any $\tilde{X} \in A(X)$.

or

$$\{\gamma(\theta) \in \Theta : y\, \Phi(\gamma(\theta))\, x\} >_S \{\gamma(\theta) \in \Theta : x\, \Phi(\gamma(\theta))\, y\} \qquad OR-2$$

We call $OR(X)$ the set of all order-restricted preference profiles on X. In words, a profile is order-restricted on X if we can order the types of the individuals in such a way that for any pair of alternatives the set of types preferring one of the alternatives all lie to one side of those who prefer the other. It is important to emphasize that the ordering of types is not conditional on the pair of alternatives under consideration, while the "cut-off" types may depend on the pair. Example 4 below illustrates the concept.

Example 4 *(Austen-Smith and Banks, 1999): Consider the preferences over $X = \{x,\ y,\ z\}$, with the order $x < y < z$, for the types $\theta_1 < \theta_2 < \theta_3$, displayed in Table 3. This profile is order-restricted over X, since there exists a permutation γ, defined by $\gamma(\theta_1) = \theta_2$, $\gamma(\theta_2) = \theta_1$ and $\gamma(\theta_3) = \theta_3$, such that under this renaming of types we have that:*

- $\{\theta :\ x\, \Phi(\theta)\, y\} = \{\theta_1, \theta_2\} <_S \{\theta_3\} = \{\theta :\ y\, \Phi(\theta)\, x\}$;
- $\{\theta :\ x\, \Phi(\theta)\, z\} = \{\theta_1, \theta_2\} <_S \{\theta_3\} = \{\theta :\ z\, \Phi(\theta)\, x\}$;
- $\{\theta :\ y\, \Phi(\theta)\, z\} = \{\theta_1\} <_S \{\theta_2, \theta_3\} = \{\theta :\ z\, \Phi(\theta)\, y\}$. $\square$

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
z	y	y
y	z	x

Table 3: Example 4

The following results exhibit the close relationship between OR and SC:

Lemma 1 *If a preference profile $\succ$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on X then, it satisfies order-restriction on X.*

Proof In order to show this, consider a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$. Choose any x, $y \in X$ and, without loss of generality, assume $y > x$. Since Θ is finite, there exists $\theta^* \in \Theta$ such that $\theta^* = \min_{\theta} \{\theta \in \Theta : y\, \Phi(\theta)\, x\}$. If such type does not exist, then $x\, \Phi(\theta)\, y$ for all $\theta \in \Theta$ and order-restriction follows immediately. Otherwise, by single-crossing, $y\, \Phi(\theta)\, x$ for all $\theta > \theta^*$. Finally, by the completeness of the binary relation, $x\, \Phi(\theta)\, y$ for all $\theta < \theta^*$. Hence,

$(\succ_1, \ldots, \succ_n) \in OR(X)$. $\square$

However, the converse is not true. Just consider the original ordering in Example 4. As we showed, it is in $OR(X)$, but it is not in $SC(X)$ as, for example, $z\,\Phi(\theta_1)\,y$ while $y\,\Phi(\theta_2)\,z$, being $z > y$ and $\theta_2 > \theta_1$. Nevertheless we have the following result:

Lemma 2 *For any profile $\succ$, derived from $\Phi : \Theta \to P(X)$, such that $\succ \in OR(X)$, there exists a permutation $\bar{\gamma} : \Theta \to \Theta$, such that the profile $\succ^{\bar{\gamma}}$, derived from $\Phi : \bar{\gamma}(\Theta) \to P(X)$, verifies $\succ^{\bar{\gamma}} \in SC(X)$.*

Proof Consider a preference profile $\succ \in OR(X)$. Since $\succ \in OR(X)$, there exists a permutation γ such that for $\gamma(\Theta)$ and any pair of alternatives $x, y \in X$, say $x < y$, we have either $OR-1$ or $OR-2$. In the latter case, consider $\theta^* \in \gamma(\Theta)$, such that $\theta^* = \min_{\theta} \{\theta \in \gamma(\Theta) : y\,\Phi(\theta)\,x\}$. Therefore, since $y\,\Phi(\theta^*)\,x$, we have that $y\,\Phi(\theta)\,x$, for any $\theta \in \gamma(\Theta)$ such that $\theta > \theta^*$. Thus, for $\bar{\gamma} = \gamma$, the profile $\succ^{\bar{\gamma}}$ is in $SC(X)$. Instead, if γ is such that for $x < y$ it verifies $OR-1$, consider a permutation $\gamma' : \gamma(\Theta) \to \gamma(\Theta)$, such that (if $|\gamma(\Theta)| = |\Theta| = m$), $\gamma'(\theta_i) = \theta_{m-i+1}$, for every $\theta_i \in \gamma(\Theta)$. This permutation just induces a reversion of the ordering in $\gamma(\Theta)$. Then, composing γ' and γ we have a permutation $\bar{\gamma}$ such that on $\bar{\gamma}(\Theta)$ we have $OR-2$ and again, $\succ^{\bar{\gamma}} \in SC(X)$. $\square$

Notice that this result amounts to an equivalence (under renaming of types in Θ) of SC and OR.

4 The Representative Voter Theorem

Single-crossing (order-restriction) has some properties that have been shown to be very useful in the analysis of collective decision-making processes. The first one, already mentioned in other parts of the paper, is that it guarantees the existence of majority voting equilibria.

Additionally, it can also be shown that, when preferences are order-restricted, the *median type* agent in the order on $(\theta_1, \ldots, \theta_n)$ (which is unique in our framework since I is odd) is decisive in all pairwise majority contests between alternatives in $\tilde{X}$, for all $\tilde{X} \in A(X)$.[10] This result is sometimes referred to as the Representative Voter Theorem (RVT) or, alternatively, as the "second version" of the Median Voter Theorem.

[10]See, for example, Rothstein (1991), Myerson (1996), Gans and Smart (1996), Austen-Smith and Banks (1999) and Persson and Tabellini (2000).

In this section we will present formally the RVT, leaving for the next section the task of proving its game-theoretic counterpart. But first, two comments are in order. The first is to note that we will present only a simplified version of the original RVT. It is simpler because neither individual indifference nor the case with an even number of voters is considered.[11]

The second observation is that the original formulation and the proof of the RVT were given in the context of order-restricted preferences (see Rothstein, 1991). However, since we have shown the equivalence, under renaming of types, of order-restriction and single-crossing, we will exploit in the next section the fact that the median choice rule is strategy-proof over single-crossing preferences to prove the validity of the RVT in strategic environments. So, to maintain the internal consistency of the paper, our proof here of the RVT uses the single-crossing condition, instead of order-restriction.

The non-strategic version of the Representative Voter Theorem is as follows:

Theorem 1 *Let $f^m : OR(X) \to X$ be the median choice rule on the domain of order-restricted preferences. Then, for each preference profile $\succ \in OR(X)$, and for every nonempty subset $\tilde{X} \in A(X)$, $f^m(\tilde{\succ}) = M(\tilde{X}, \tilde{\Phi}(\theta_r))$, where $\theta_r = m^n(\theta_1, \theta_2, \ldots, \theta_n)$.*

Proof. Consider a preference profile $\succ \in OR(X)$. By Lemma 2, there exists a profile $\succ^{\bar{\gamma}} \in SC(X)$ that obtains by renaming the types $\{\theta_i\}_{i \in I}$. Take the agenda $\tilde{X} \in A(X)$ and the restriction of $\succ^{\bar{\gamma}}$ to $\tilde{X}$, $\tilde{\succ}^{\bar{\gamma}}$. Define the set of individuals' maximal alternatives in $\tilde{X}$ according to $\tilde{\succ}^{\bar{\gamma}}$ as follows: $T(\tilde{X}, \tilde{\succ}^{\bar{\gamma}}) = \{\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_i^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})\}$. We claim that, for all $i, j \in I$, if $\theta_i^{\bar{\gamma}} < \theta_j^{\bar{\gamma}}$, then $\tau(\tilde{\succ}_i^{\bar{\gamma}}) < \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Suppose not. That is, assume by contradiction $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \geq \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Since $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \tilde{\succ}_i^{\bar{\gamma}} \tau(\tilde{\succ}_j^{\bar{\gamma}})$ and $\theta_i^{\bar{\gamma}} < \theta_j^{\bar{\gamma}}$, by single-crossing, we have that $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \tilde{\succ}_j^{\bar{\gamma}} \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Absurd. Thus, the set $T(\tilde{X}, \tilde{\succ}^{\bar{\gamma}})$ has to be ordered from the lowest to the highest top; and, therefore, it follows that $f^m(\tilde{\succ}^{\bar{\gamma}}) = m^n(\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})) = \tau(\tilde{\succ}_r^{\bar{\gamma}}) = M(\tilde{X}, \tilde{\Phi}(\theta_r^{\bar{\gamma}}))$, where $\theta_r^{\bar{\gamma}} = m^n(\theta_1^{\bar{\gamma}}, \theta_2^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}})$. Finally, notice that $\theta_r^{\bar{\gamma}} = \theta_r$, where $\theta_r = m^n(\theta_1, \ldots, \theta_n)$, since, according to the proof of Lemma 2, $\bar{\gamma}$ is either the identity (meaning that, for each i, $\theta_i^{\bar{\gamma}} = \theta_i$) or it is a reversion of the original ordering (implying that, for each i, $\theta_i^{\bar{\gamma}} = \theta_{m-i+1}$). In either case, $m^n(\theta_1^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}}) = m^n(\theta_1, \ldots, \theta_n)$. $\square$

In words, Theorem 1 says that, given any subset of policies $\tilde{X} \in A(X)$, the alternative chosen by a society with order-restricted preferences is the

[11] For a more complete treatment, see the references listed in footnote 10.

most preferred option of the median type agent.[12] This result holds also under single-peakedness if X is the range of f^m, but not necessarily in other cases. Figure 1 below illustrates this point:

[Insert Figure 1 about here]

In the picture, preferences over the full set of alternatives, $X = [0, 1]$, are single-peaked. Therefore, the Median Voter Theorem applies, and agent 2's *unrestricted* top, τ_2, wins in pairwise majority voting. Moreover, the *induced* profile of preferences over the subset $\tilde{X} = \{a, b, c, d\} \subset X$ satisfies also single-peakedness, (along the linear ordering $c < a < b < d$).[13] However, it is not single-crossing. Then, it turns out that agent 2's most preferred alternative in $\tilde{X}$, d, is defeated by the alternative b, which is agent 3's *restricted* top and the Condorcet winner in $\tilde{X}$.

Thus, what this example shows is that under single-peakedness the median agent may depend on the particular agenda considered. This does not happen under single-crossing. Theorem 1 guarantees that the median type θ_r (and hence the individual who is of this type) is decisive over any non-empty subset $\tilde{X} \in A(X)$.

However, is the collective outcome predicted by the RVT robust to individual or group manipulation? That is, can we expect this outcome to hold when voters act strategically? The Representative Voter Theorem is a result derived under the assumption that individuals honestly reveal their preferences or, alternatively, under the assumption that the decision-maker knows them. Both assumptions are obviously very strong.

Fortunately, it turns out that, even if we relax these assumptions, admitting both private information of individual values and strategic behavior on the part of voters, the RVT still holds. As we will see in the next section, the reason is that the median choice rule f^m is strategy-proof on the domain of single-crossing preference profiles. This implies that, in any majority contest, each agent has a dominant strategy, which is to honestly reveal his preferences. Therefore, the RVT applies, meaning that the outcome predicted by Theorem 1 must be expected no matter what strategic considerations are allowed. In the following section, we derive this result formally and we pro-

[12] Rothstein (1991) has also shown that, when preferences are strict and the number of voters is odd, as in our case, the preference ordering induced by the majority rule coincides with the preference relation of the median type agent. This implies that the majority preference relation inherits all the properties of the median type agent's preference ordering. In particular, transitivity. Gans and Smart (1996) have proven a similar result for non-strict preference orderings, but under strict single-crossing.

[13] In fact, this is the profile introduced in Example 3 above.

vide an *indirect* mechanism that implements the prediction of the RVT in dominant strategies.

5 Manipulation over single-crossing domains

The manipulation of the median rule has been studied for a long time in the literature of social choice. The earliest reference goes back to the seminal paper of Black (1948). Since then, a lot of progress has been made towards the understanding of its properties. For instance, it is well-known today that there exists a preference domain where this voting procedure performs quite well, in terms of its capacity to extract truthful information about the preferences of the agents. This domain is of course single-peakedness.

In this section, we analyze whether the median choice rule can be manipulated on a different preference domain, namely over single-crossing preferences. Even though this family of preferences is now largely used in models of collective decision-making process, nothing is said in the existence literature about the possibility of manipulation over this domain. In particular, people uses the "single-crossing version" of the Median Voter Theorem without asking about its strategic foundation. The main purpose here is therefore to fill this gap.

Our main result is the following:

Proposition 1 *The median choice rule f^m is strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*

Proof Consider a profile $\tilde{\succ} = (\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, where agent i, of type θ_i, has preferences $\tilde{\succ}_i$. Suppose that there exists another type $\hat{\theta}_i$ such that $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) \,\tilde{\succ}_i\, f^m(\tilde{\succ})$. Furthermore, without loss of generality, assume that $\tau(\tilde{\succ}_i) < f^m(\tilde{\succ})$. We have two cases to consider:

1. $\tau(\hat{\succ}_i) \leq f^m(\tilde{\succ})$. Then, $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Contradiction;
2. $\tau(\hat{\succ}_i) > f^m(\tilde{\succ})$. Then $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Let us call $\tilde{\tau} = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ and $\hat{\tau} = f^m(\hat{\succ}_i, \tilde{\succ}_{-i})$. Since we assume that $\tilde{\succ}$ verifies the single-crossing property, we have that $\hat{\tau} \,\tilde{\Phi}(\theta)\, \tilde{\tau}$ for all $\theta \geq \theta_i$. On the other hand, since $\tilde{\tau}$ is the maximal for at least one $\tilde{\succ}_j$ in $\tilde{\succ}$, it must be that the type corresponding to $\tilde{\succ}_j$, say θ_j, is such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{\tau}$, by single-crossing we have that $\tilde{\tau} \,\tilde{\Phi}(\theta)\, \tau(\tilde{\succ}_i)$ for every $\theta > \theta_j$. In particular for θ_i. Contradiction. □

Thus, Proposition 1 makes the important contribution of proving that, apart from single-peakedness, there exists another very *natural* preference domain over the real line where strategy-proof choice rules can be found. That is, it shows that single-crossing preferences constitute a domain restriction that allows not only majority voting equilibria, but also the existence of non-trivial strategy-proof social choice functions. In particular, this is true for the median choice rule.[14]

Since single-crossing preferences are not necessarily single-peaked (see, for instance, Example 2 in the text), this result has the important implication that the violation of single-peakedness does not preclude the existence of non-manipulable social choice functions over the real line.

Moreover, single-crossing not only may fail to satisfy single-peakedness, but also it implies that individual preferences may be correlated. Therefore, Proposition 1 also proves that the absence of independent individual preference domains is not an impediment either to find strategy-proof rules. At least for some non-trivial and common decision rules, the existence of a linear ordering of the types of the agents (with the requirement already mentioned that, for any pair of alternatives, the set of types who prefer more one of the alternatives all lie to one side of those who prefer more the other) is a sufficient condition that ensures non-manipulation at the individual level. Furthermore, as the following proposition shows, it turns out that it also guarantees non-manipulation at group level:

Proposition 2 *The median choice rule f^m is group-strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*

Proof. Consider a profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$, with associated types $(\theta_1, \ldots, \theta_n)$. Suppose there exists a coalition $D \subseteq I$ and a list of alternative types for members of D, $(\hat{\theta}_i)_{i \in D}$, $(\hat{\theta}_i)_{i \in D} \neq (\theta_i)_{i \in D}$, such that the joint declaration generated by $\hat{\theta}_D$, $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, produces a preferred social outcome for every member of the coalition. That is, for all $i \in D$,

$$f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) \;\tilde{\succ}_i\; f^m(\tilde{\succ}_D, \tilde{\succ}_{D^c}),$$

[14]Tohmé and Saporiti (2003) shows that the whole family of *tops-only*, *efficient* and strategy-proof social choice functions over single-crossing preferences is given by a subclass of the *extended median rules*, obtained by distributing the *phantom voters* at the extremes of the non-negative real line. This subclass, where each phantom voter is either a *leftist* or a *rightist*, is sometimes referred to as *positional dictators* (see Moulin (1988), pp. 302). These rules select the kth ranked peak among the tops of the reported preference orderings, for some $k = 1, \ldots, n$. For example, if $k = 1$, we have the *leftist rule*, which chooses the smallest reported peak of a real voter. Of course, the median choice rule is also a particular case.

where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$. For simplicity, call $f^m(\tilde{\succ}) = \tilde{\tau}$ and $f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) = \hat{\tau}$. Notice that, by the definition of f^m, $\tilde{\tau}$ and $\hat{\tau}$ coincide with the tops corresponding to the orderings reported by some voters. Denote these agents j and $j^{'}$ and their types θ_j and $\theta_{j'}$, respectively. Since $\tilde{\tau} \neq \hat{\tau}$ assume that $\tilde{\tau} < \hat{\tau}$. Then, for all $i \in D$, $\tau(\tilde{\succ}_i) > \tilde{\tau}$. Suppose not. That is, assume $\tau(\tilde{\succ}_i) \leq \tilde{\tau}$ for some agent i in D. If $\tau(\tilde{\succ}_i) = \tilde{\tau}$, then $\tilde{\tau} \tilde{\succ}_i \hat{\tau}$, which contradicts our hypothesis. Consider, instead, that $\tau(\tilde{\succ}_i) < \tilde{\tau}$. Since $\hat{\tau} \tilde{\succ}_i \tilde{\tau}$, by single-crossing we have that for all $\theta > \theta_i$, $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tilde{\tau}$. Then, θ_j has to verify that $\theta_j < \theta_i$ and, by single-crossing, $\tilde{\tau}\, \tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$ implies $\tilde{\tau}\, \tilde{\Phi}(\theta_i)\, \tau(\tilde{\succ}_i)$. Contradiction. Then, $\tau(\tilde{\succ}_i) > \tilde{\tau}$, for all $i \in D$. The rest of the proof is as follows. By definition,

$$f^m(\tilde{\succ}_D, \tilde{\succ}_{D^c}) = m^n(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)) = \tilde{\tau},$$

while

$$f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) = m^n(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c}) = \hat{\tau}.$$

Two cases are possible:

1. For each $i \in D$, $\tau(\hat{\succ}_i) > \tilde{\tau}$. Then $\hat{\tau} = \tilde{\tau}$. Contradiction.

2. For some $i \in D$, $\tau(\hat{\succ}_i) \leq \tilde{\tau}$. Then, by rewritten $(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c})$ as $(y_1, \ldots, y_n)$, we have that

$$\Big| \{j \in \{1, \ldots, n\} : y_j \leq \tilde{\tau}\} \Big| \geq \frac{(n+1)}{2}.$$

   But this implies that $m^n(y_1, \ldots, y_n) \leq \tilde{\tau}$. That is, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \leq f(\tilde{\succ}_D, \tilde{\succ}_{D^c})$, which contradicts our initial hypothesis. $\square$

Next we will use these positive results for the median choice rule to provide the game-theoretic counterpart of the Representative Voter Theorem. To do that, notice first that, according the the Revelation Principle, if a social choice function is *truthfully* implementable in a dominant strategy equilibrium, it must be strategy-proof. That is, strategy-proofness is a *necessary* condition for truthfully or *direct* implementation.

However, it is not *sufficient*. It is in fact sufficient when the preference domain of the social choice function can be written as a Cartesian product (Moore, 1992). Otherwise, the direct revelation mechanism is not well-defined, in the sense that the set of strategies of each agent, i.e., the set of admissible individual preference orderings that can be declared, depends on the strategies used by the others.[15]

This is precisely our case. Proposition 1 shows that f^m is strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$. Thus, the necessary condition for the application of the Revelation Principle holds. However, under single-crossing, individual preferences may be correlated. Therefore, $SC(\tilde{X})$ cannot be written as a Cartesian product subset of $P(\tilde{X})^n$. That is, the sufficient condition fails, and the implementation of f^m in dominant strategy equilibria has to be explicitly analyzed.

In what follows, we will informally present an extensive game form that can be used to *indirectly* implement f^m in dominant strategies. After that, we will argue that this game form is essentially equivalent to a *reduced* mechanism in normal form, and we prove that this last mechanism succeeds in implementing the median rule. We will also briefly discuss why the extensive game form or its associated reduced game form works, but not the direct mechanism in which each individual simply declares his top in $\tilde{X}$. Finally, we will derive as a by-product the game-theoretic equivalent of Proposition 1.

5.1 Implementation of the median choice rule

Suppose individuals in I have preferences $(\succ_1, \ldots, \succ_n) \in SC(X)$. Assume the selection of a social outcome in $\tilde{X}$, which is the planner's basic problem, is indirectly performed by the following two-stage voting procedure. In the first stage, individuals select by pairwise majority voting a *representative* individual from the set I. Then, in the second stage, the winner chooses an alternative in $\tilde{X}$, which is then the policy implemented by the planner.

Since in the last stage each individual i has a dominant strategy, which is simply to choose his most preferred alternative in $\tilde{X}$, $\tau(\tilde{\succ}_i)$, it is immediate to see that this extensive game form is equivalent to a *reduced* strategic game form in which individuals choose by pairwise majority comparisons an alternative in the set $T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_i), \ldots, \tau(\tilde{\succ}_n)\}$.

Now we prove that this reduced mechanism can be used to implement f^m in a dominant strategy equilibrium.

Definition 6 *A mechanism Γ with consequences in $\tilde{X}$ is a strategic game form $\langle I, (S_i), \phi\rangle$ where, for each $i \in I$, S_i is the set of actions available*

[15] A possible way of solving this consists in asking to each individual to report a preference profile, instead of his individual preference ordering. If the social choice function is strategy-proof, then it can be shown that reporting the true preferences of the whole society is a dominant strategy for each individual. See Osborne and Rubinstein (1994) for a formal proof.

for agent i, and $\phi : \prod_{i \in I} S_i \to \tilde{X}$ is an outcome function that associates an alternative with every action profile.

We say that Γ implements a social choice function $f : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium for the mechanism, yielding the same outcome as f for each possible preference profile $\tilde{\succ} \in SC(\tilde{X})$. This is formally stated in Definition 7.

Definition 7 *The mechanism $\Gamma = \langle I, (S_i), \phi \rangle$ implements the social choice function $f : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$, such that $\phi(s^*(\tilde{\succ})) = f(\tilde{\succ})$ for all $\tilde{\succ} \in SC(\tilde{X})$.*

Proposition 3 *There exists a mechanism that implements $f^m : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof. Consider a preference profile $\tilde{\succ} \in SC(\tilde{X})$ and the mechanism $\Gamma = \langle I, (S_i), \phi \rangle$, where I is the set of players; an action for agent $i \in I$ is simply to choose an element in $S_i = T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_i), \ldots, \tau(\tilde{\succ}_n)\}$; and the outcome function $\phi(s_1, \ldots, s_n) = m^n(s_1, \ldots, s_n)$. We will show that the action profile $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ constitutes a dominant strategy equilibrium of the game induced by Γ. That is,

$$\phi(s_1, \ldots, \tau(\tilde{\succ}_i), \ldots, s_n) \;\tilde{\succ}_i\; \phi(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

for all i, $\hat{s}_i \neq \tau(\tilde{\succ}_i)$, $s_{-i} \in \prod_{j \neq i} S_j$. Since, by definition, $\phi(\cdot) = m^n(\cdot)$, we can easily recast the proof of Proposition 1 to fit in this scheme. Suppose that there exists such $\hat{s}_i$. Call $\tilde{s} = \phi(\tau(\tilde{\succ}_i), s_{-i})$ and $\hat{s} = \phi(\hat{s}_i, s_{-i})$. Without loss of generality, assume $\tau(\tilde{\succ}_i) < \tilde{s}$. We have two cases to consider:

1. $\hat{s}_i \leq \tilde{s}$. Then, $m^n(\tau(\tilde{\succ}_i), s_{-i}) = m^n(\hat{s}_i, s_{-i})$ and, therefore, $\phi(\tau(\tilde{\succ}_i), s_{-i}) = \phi(\hat{s}_i, s_{-i})$. Contradiction.

2. $\hat{s}_i > \tilde{s}$. Then the new median $\hat{s}$ will be in the interval $[\tilde{s}, \hat{s}_i]$. By hypothesis, $\hat{s} \,\tilde{\succ}_i\, \tilde{s}$. Furthermore, since the preferences are single-crossing on $T(\tilde{X}, \tilde{\succ})$ and $\hat{s} > \tilde{s}$, for every $\theta > \theta_i$ we have that $\hat{s} \,\tilde{\Phi}(\theta)\, \tilde{s}$. On the other hand, notice that, since each $S_j = T(\tilde{X}, \tilde{\succ})$, there must exist $\theta_j \in \Theta$ such that $\tilde{s} = \tau(\tilde{\Phi}(\theta_j))$. Moreover, θ_j must be such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{s}$ and $\tilde{s} \,\tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$, by single-crossing, we have that $\tilde{s} \,\tilde{\Phi}(\theta)\, \tau(\tilde{\succ}_i)$ for all $\theta > \theta_j$; in particular for θ_i. Contradiction.

Therefore, $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ is a dominant strategy equilibrium. $\square$

The fact that the alternative declared by each agent is restricted to belong to $T(\tilde{X}, \tilde{\succ})$, the set of all individual maximal alternatives in $\tilde{X}$, is crucial for the proof of Proposition 3. It is easy to see that a mechanism based on direct declarations of the most preferred alternatives in $\tilde{X}$ cannot be used to implement f^m. For instance, in Example 2, if agents are asked to declare their most preferred alternatives in $\tilde{X} = \{x, y, z\}$, then manipulation cannot be avoided: if agent 1 and agent 3 declare y and z, respectively, then player 2 will prefer to announce z instead of his true top x.[16]

Instead, the reason of why our *indirect* mechanism works is because the induced preferences over the set $T(\tilde{X}, \tilde{\succ})$, derived from $\tilde{\succ} \in SC(\tilde{X})$, are single-peaked. This is formally shown in Lemma 3 below. Notice that $T(\tilde{X}, \tilde{\succ})$ can be identified with the set of *actual* ideal points, $\{x \in \tilde{X} : \exists i \in I \text{ such that } x = \tau(\tilde{\succ}_i)\}$:

Lemma 3 *If a preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ is single-crossing over $\tilde{X}$, then the restriction of $\tilde{\succ}$ over the set $T(\tilde{X}, \tilde{\succ})$ is single-peaked.*

Proof. For a given profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ and the associated set $T(\tilde{X}, \tilde{\succ})$, consider the restriction of $\tilde{\succ}$ to $T(\tilde{X}, \tilde{\succ})$, denoted $\tilde{\succ}^T = (\tilde{\succ}_1^T, \ldots, \tilde{\succ}_n^T)$. By contradiction, suppose $\tilde{\succ}^T \notin SP(T)^n$, where $SP(T)^n$ is the set of all single-peaked preference profiles over $T(\tilde{X}, \tilde{\succ})$ (with respect to the linear order $\leq$). Then, there exist an individual $i \in I$, with type $\theta_i \in \Theta$, and x, y, $\tau(\tilde{\succ}_i) \in T(\tilde{X}, \tilde{\succ})$ such that

$$x < y \leq \tau(\tilde{\succ}_i), \text{ but } x \tilde{\succ}_i^T y.$$

Thus, $y \neq \tau(\tilde{\succ}_i)$. Moreover, since $\tilde{\succ}^T \in SC(T)$, $x \tilde{\succ}_j^T y$ for all $\theta_j \leq \theta_i$. This means $y \neq \tau(\tilde{\succ}_j)$ for all $\theta_j \in \{\theta_1, \theta_2, \ldots, \theta_i\}$. However, since we assume $y \in T(\tilde{X}, \tilde{\succ})$, then $y = \tau(\tilde{\succ}_k)$ for some individual $k \in I$ with type $\theta_k \in \{\theta_{i+1}, \theta_{i+2}, \ldots, \theta_n\}$. Then, $y \tilde{\succ}_k \tau(\tilde{\succ}_i)$ implies $y \tilde{\succ}_j \tau(\tilde{\succ}_i)$ for all $\theta_j \leq \theta_k$. In particular, for θ_i. Contradiction. The same argument applies if $\tau(\tilde{\succ}_i) \leq y < x$ and $x \tilde{\succ}_i^T y$. Hence, $\tilde{\succ}^T \in SP(T)^n$. $\square$

[16] Proposition 1 shows that individual manipulation is ruled out when agents are required to declare a complete preference ordering, and not just the top alternative. The intuition is again illustrated by Example 2. Notice that in this case individual 1 cannot summit an ordering with the alternative y as its top without violating the single-crossing condition. Thus, player 2 has no reason to lie.

It is easy to show that the converse of Lemma 3 does not hold. That is, preferences can be single-peaked over $T(\tilde{X}, \tilde{\succ})$, but not necessarily single-crossing on $T(\tilde{X}, \tilde{\succ})$. The preference profile presented in Table 4 below provides an example in which this happens.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
w	x	y	z
x	y	x	y
y	z	w	x
z	w	z	w

Table 4: Counterexample

Finally, we derive the following Corollary from Proposition 3:

Corollary 1 *For any $\tilde{X} \in A(X)$, there exists a mechanism that implements $f^m : OR(\tilde{X}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof. Trivial. Consider any preference profile $\tilde{\succ} \in OR(\tilde{X})$. By Lemma 2, there exists a permutation $\bar{\gamma}$ of Θ that generates a profile $\tilde{\succ}^{\bar{\gamma}} \in SC(\tilde{X})$. Hence, the mechanism defined in Proposition 3 yields, as the outcome of its dominant strategy equilibrium, the median value of the maximal alternatives over $\tilde{X}$, $\phi(\tilde{\succ}^{\bar{\gamma}}) = m^n(\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})) = \tau(\tilde{\Phi}(\theta_r^{\bar{\gamma}}))$. Finally, this outcome coincides with $f^m(\tilde{\succ})$ because, as seen in Proposition 1, $m^n(\theta_1^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}}) = m^n(\theta_1, \ldots, \theta_n)$. $\square$

This Corollary provides the *strategic counterpart* of Proposition 1. That is, it shows that, when preferences are order-restricted, the social outcome under pairwise majority voting, i.e. the most preferred alternative of the median type, can be attained by a reduced mechanism in which agents are allowed to declare one of the individual maximal alternatives in the feasible set of policies. Or, alternatively, it can be achieved by following a two-stage voting procedure in which, first, the individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

6 Final remarks

In this paper, we exhibited several results. First of all, we have proven that, apart from single-peakedness, there exists another very *natural* preference domain over the real line for which strategy-proof choice rules can be found. Concretely, we have shown that single-crossing preferences constitute a domain restriction that allows not only majority voting equilibria, but also the existence of non-trivial strategy-proof (as well as group-strategy-proof) social choice functions. In particular, this is true for the median choice rule.

The first feature to remark of this result is that single-crossing preferences do not necessarily satisfy single-peakedness. But, as it is known, in one-dimensional collective decision models this is one of the most frequently applied domain restrictions that guarantee strategy-proofness. Thus, the result found here shows that the violation of single-peakedness does not preclude the existence of non-manipulable social choice functions over the real line.

Furthermore, single-crossing also implies that individual preferences are correlated. Therefore, Proposition 1 also proves that the absence of independent individual preference domains is not an obstacle for the existence of strategy-proof rules. At least for some non-trivial and common decision rules, the existence of a certain kind of linear ordering of the types of the agents is a sufficient condition that ensures non-manipulation both at the individual and at the group level.

Another important results are summarized in Lemmas 1 and 2, which exhibit the close relation between single-crossing and order-restriction. A previous work in the same direction is Gans and Smart (1996), in which these preference domains are shown to be essentially equivalent. Nevertheless, our results differ from theirs in two ways. First, ours seem to be more consistent with Rothstein's original characterization of order-restriction. Second, particular attention is devoted here to the fact that these conditions may not be *directly* equivalent. The crucial point to understand this difference is that, unlike single-crossing, order-restriction does not assumes any ordering on the set of possible alternatives. Furthermore, it is precisely this feature that makes order-restriction so interesting for analyzing strategy-proofness in multi-dimensional choice spaces and over restricted agendas.

Finally, these previous results are used at the end of the paper to show that the Representative Voter Theorem has a well-defined non-cooperative strategic foundation. Concretely, we show that the collective outcome predicted by this Theorem can be implemented through a simple sequential mechanism in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner. Given that the structure of this mechanism presents some features that we observe frequently in "real" voting processes, the analysis carried out here may also provide insights for a rationale of these "real" voting situations.

At the same time, there are significant topics that this paper does not cover. The most important task that we have left for future work is to fully characterize the family of strategy-proof social choice functions over single-crossing preference profiles. Of course, the classes that also satisfy other requirements like anonymity, Pareto efficiency or combinations of them should also be determined.

The second relevant aspect that we do not address here is how these results change when individual preference orderings are allowed to express indifference between different alternatives. Clearly, our simplification is justified by the fact that the set of possible social outcomes is finite. However, we guess substantial changes may be expected in our results if this assumption is dropped.

Finally, another problem that must be answered is how to extend single-crossing and order-restriction to multidimensional spaces. That is, we should consider the way in which these preference restrictions can deal with both multidimensional choice sets and political conflicts of interests that cannot be projected onto a one-dimensional space.

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